DELTA FUNCTION IN ELECTROSTATICS

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In the Electrostatics section of the university general physics courses [1], [2] students study Gauss's Law, which is of great theoretical importance and is often is used to determine the electric field strength in the cases of high symmetric charge distribution, including the field of a point charge. According to the Gauss low:

$$\int_V \operatorname{div}\vec{E}\,\mathrm{d}V = \oint_S \vec{E}\,\mathrm{d}\vec{S}\,, \tag{1}$$

where S is the surface enclosing the volume V.

In the case of a point charge q, the field has radial symmetry; therefore, it is convenient to choose a closed surface in the form of a sphere of radius r, in the center of which the charge is located. Then,

$$\int_S \vec{E}\,\mathrm{d}\vec{S} = \int_S \vec{E}\vec{n}\,\mathrm{d}S = \int_S E(r)\,\mathrm{d}S = E(r)\int_S \mathrm{d}S = 4\pi r^2 E(r) \tag{2}$$

.

According to Maxwell's equation (rather, Gauss's law)

$$\varepsilon_0 \int_S \vec{E}\,\mathrm{d}\vec{S} = q\,, \tag{3}$$

and substituting the flux expression (2) in (3) we obtain

$$\varepsilon_0 E(r) 4\pi\varepsilon_0 r^2 = q \quad \Rightarrow \quad E(r) = \frac{q}{4\pi\varepsilon_0 r^2}\,, \tag{4}$$

i.e. Coulomb's law.

On the other hand, according to (1), to calculate the flux, one can use the calculation $\operatorname{div}\vec{E}$ followed by integration over volume. In spherical coordinates

$$\operatorname{div}\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(E_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi} \tag{5}$$

.

The derivatives with respect to the angular coordinates are equal to zero due to spherical symmetry, so

$$\operatorname{div}\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right).$$

Substituting here expression (4), we obtain

$$\operatorname{div}\vec{E} = \frac{1}{4\pi\varepsilon_0 r^2}\frac{\partial}{\partial r}\left(r^2\frac{q}{r^2}\right) = 0. \tag{6}$$

Thus, such an “elementary”, “simple” case of a point charge leads to a contradiction: on the one hand, the right side of the Gauss theorem is not equal to zero (3), and on the other hand, in (6) it is equal to zero. Of course, it is possible to extend the dependence *E*(r) by the condition $r \neq 0$, but in this case, (6) holds in all areas except $r = 0$. But for $r = 0$, the function *E*(r) is undefined and the validity of the Gauss theorem is unclear.

The purpose of this work is to find the correct form of writing the expression for the field strength of a point charge at $r\in(0, \infty)$.

To achieve this goal, it is necessary to redefine the function *E*(*r*), however, in this case, one will have to go beyond the “ordinary” functions, turning to generalized ones.

Let’s use a dimensionless function $\theta(r)$, which is defined for all values $r \geq 0$, such that $\theta(r > 0) = 0$, and its derivative is equal to the Dirac delta-function $\delta(r)$:

$$\frac{d\theta}{dr} = \delta(r). \tag{7}$$

Let us extend the function (4) (defined only for $r > 0$) by its value at the point $r = 0$:

$$E(r) = \frac{q}{4\pi\varepsilon_0 r^2}\left(1+\theta(r)\right). \tag{8}$$

At all points except zero, this function coincides with (4).

Now, let’s find the divergence of the field (8) in spherical coordinates (taking into account its spherical symmetry)

$$\operatorname{div}\vec{E} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 E_r\right) = \frac{1}{4\pi\varepsilon_0 r^2}\left(\frac{d}{dr}\left(r^2\frac{q}{r}\right) + q\frac{d\theta}{dr}\right) = \frac{q}{4\pi\varepsilon_0 r^2}\delta(r) \tag{9}$$

.

From (9) it’s clear that the divergence of the electric field of a point charge is no longer equal to zero, and integrating the resulting expression over the volume

$$\int_V \operatorname{div}\vec{E}\,dV = \int E_r 4\pi r^2\,dr = 4\pi\frac{q}{4\pi\varepsilon_0}\int\delta(r)\,dr = \frac{q}{\varepsilon_0}$$

we obtain expression (3), as it should be.

Conclusions. For the values of $r \in (0,\infty)$ the correct form of writing the Coulomb law and the field strength of a point charge requires the use of generalized functions, in particular, the Dirac delta function.

The prospects for further research we see in the application of the obtained results in the lectures course when students study the chapter “Electrostatics”.

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